St.Philomena's College (Autonomus), Mysore

PG Department of Mathematics

Question Bank (Revised Curriculum 2018 onwards)

Second Year - Fourth Semester (2018 -20 Batch)

Course Title (Paper Title): Linear Algebra II Q.P.Code-57104

Unit	S.No	Question	Marks
1	1	Define inner product space.	2m
1	2	Define norm and find the norm of (4,5,6).	2m
1	3	Define an orthogonal complement. Find the orthogonal compliment of y-axis.	2m
1	4	If V be a vector space of polynomials with inner product given by $(f,g) = \int_0^1 f(t)g(t)dt$ then find (f,g) if $f(t) = t+2$ and $g(t) = t^2 - 2t - 3$.	2m
1	5	If V be a vector space of polynomials with inner product given by $(f,g) = \int_0^1 f(t)g(t)dt$ then find $\|f\|$ if $f(t) = t^3 - 3$.	2m
1	6	In any inner product space show that $\|\alpha u\| = \|\alpha\|\|u\| \ \forall \alpha \in F \ u \in V$	2m
1	7	Prove that absolute value of cosine of angle is almost one.	2m
1	8	Define orthogonal set and orthonormal set.	2m
1	9	Obtain an orthonormal basis with respect to the standard inner product for the subspace of R^3 is generated by $(1,0,3)$ and $(2,1,1)$.	2m

1	10	If V is a finite dimensional inner product space and W is a subspace of V then prove that $(W^{\perp})^{\perp} = W$	2m
2	11	Define normal linear operator.	2m
2	12	Consider the inner product space R^4 with the standard inner product. If $u = (3, 2, k, -5)$ and $v = (1, k, 7, 3)$ are orthogonal. Find the value of k.	2m
2	13	If u and v are vectors in an inner product space V, then prove that $\|u + v\|^2 + \|u - v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$.	2m
2	14	Define Hermitian adjoint of T and prove that $(T^*)^* = T$.	2m
2	15	Define unitary transformation.	2m
2	16	If $T \in A(V)$ is unitary then prove that $TT^* = I = T^*T$.	2m
2	17	Prove that T is unitary then prove that $\|Tv\| = \|v\|$.	2m
2	18	If T is unitary and λ is a characteristic root of T, then prove that $\lvert\lambda\rvert = 1$.	2m
3	19	Find the symmetric matrix A associated with $q(x, y) = 2x^2 + xy + y^2$	2m
3	20	Define bilinear form.	2m
3	21	Define symmetric and skew-symmetric bilinear form.	2m
3	22	Find the rank and signature of the quadratic form $x_1^2 - 4x_1x_2 + x_2^2$.	2m
3	23	Show that $f(u, v) = f_1(u)f_2(u)$ is a bilinear form.	2m
3	24	Show that $f(x, y) = x_1y_1 + x_2y_2 + \cdots + x_ny_n$ is a bilinear form over $V_n(F)$ where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$.	2m

4	25	Define a minimal polynomial for T over F.	2m
4	26	Define eigen value and eigen vector of T.	2m
4	27	Determine the rank and signature of the matrix $A = \begin{pmatrix} 4 & 5 & -2 \\ 5 & 1 & -2 \\ -2 & -2 & 3 \end{pmatrix}$.	2m
4	28	Find the minimal polynomial of the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -9 \\ 0 & 1 & 6 \end{pmatrix}$.	2m
4	29	Reduce to Jordan form the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	2m
4	30	Reduce to triangular form, the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$	3m
2	31	Let $T \in A(V)$ be a Skew - Hermitian transformation then prove that all the eigen values of T are purely imaginary.	4m
2	32	If $T \in A(V)$ and if $(Tv, Tv) = (v, v)$ for all $v \in V$ then prove that T is unitary.	4m

2	33	Let T be a self adjoint operator on a finite dimensional inner product space V. Then prove that every eigen values of T are real.	4m
2	34	If λ is a characteristic root of the normal transformation N and if $vN = \lambda v$ then prove that $vN^* = \overline{\lambda} v$.	4m
2	35	If $T \in A(V)$ is such that $(vT, v) = 0$ for all $v \in V$, then prove that $T = 0$.	4m
3	36	Let f be the bilinear form on $U = \mathbb{R}^2$ and $V = \mathbb{R}^3$ that is defined by $f((x_1, x_2), (y_1, y_2, y_3)) = 2x_1y_2 - x_1y_3 + 2x_2y_1 + x_2y_2 + x_2y_3$. i). Write the matrix A of f relative to $\mathcal{A} = \{(1,0),(0,1)\}$ and $\mathcal{B} = \{(1,0,0),(0,1,0),(0,0,1)\}$. ii). Use the matrix A to compute the value of $f((3,-1),(0,4,-1))$.	4m
4	37	If $T, S \in A(V)$ and if S is regular, then T and STS^{-1}have the same minimal polynomial.	4m
2	38	If $v_1, v_2, \ldots, v_n$ is an orthonormal basis of V and if the matrix of $T \in A(V)$ in this basis is (α_{ij}), then prove that the matrix of T^* in this basis is (β_{ij}), where $(\beta_{ij}) = \overline{(\alpha_{ij})}$.	5m
2	39	Prove that the normal linear transformation N is Hermitian if and only if its characteristic roots are real.	5m
2	40	Prove that the linear transformation T on V is unitary if and only if it takes an orthonarmal basis of V into an orthonormal basis of V.	5m

2	41	Let $T \in A(V)$ be a unitary transformation and let λ be an eigen value of T then show that $\|\lambda\| = 1$.	5m
1	42	Show that any orthonormal set is linearly independent.	6m
1	43	Let V be the set of all continuous real valued function defined on the closed interval $[0,1]$ Show that V is a real inner product space with the inner product defined by $(f,g) = \int_0^1 f(t)g(t)dt$.	6m
2	44	Suppose S and T are linear operator on an inner product space V and c is a scalar. If S and T possess adjoint operator. Prove that $S+T$, cT, ST possess adjoint.	6m
3	45	Find the rank and signature of the quadratic form $x_1^2 + 2x_1x_2 + x_2^2$.	6m
3	46	Let f be the bilinear form on $U = \mathbb{R}^3$ and $V = \mathbb{R}^2$ that is defined by $f((x_1,x_2,x_3),(y_1,y_2)) = -7x_1y_1 - 10x_1y_2 - 2x_2y_1 - 3x_2y_2 + 12x_3y_1 + 17x_3y_2$; i). Write the matrix A of f relative to $\mathcal{A} = \{(1,0,0),(1,1,0),(1,1,1)\}$ and $\mathcal{B} = \{(1,-1),(2,-1)\}$. ii). Use the matrix A to compute the value of $f((2,3,1),(0,-1))$.	6m
4	47	Prove that congruence is an equivalence relation.	6m

4	48	Define the Jordan canonical form with suitable example which contains at least two Jordan blocks. Find the Jordan canonical form for the matrix: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}$	6m
1	49	Let V be a finite dimensional inner product space. Let W be a subspace of V. Prove that V is direct sum of W and $W^{\perp}$ that is $V = W \oplus W^{\perp}$.	7m
1	50	Let V be the set of real valued function $y = f(x)$ satisfying $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$. i). Prove that V is a three dimensional real vector space. ii). In V define $(u, v) = \int_{-\infty}^{0} uvdx$, find an orthonormal basis for V.	7m
1	51	State and prove the Bessal's inequality.	7m
1	52	If $u, v \in V$ and $\alpha, \beta \in F$ then show that $(\alpha u + \beta v, \alpha u + \beta v) = \alpha\overline{\alpha}(u, u) + \alpha\overline{\beta}(u, v) + \beta\overline{\alpha}(v, u) + \beta\overline{\beta}(v, v)$	7m
1	53	Let V be the set of real valued function $y = f(x)$ satisfying $\frac{d^2y}{dx^2} + 4y = 0$. Prove that V is a two dimensional real vector space. Define $(u, v) = \int_{0}^{\Pi} uvdx$, Find an orthonormal basis in V.	7m
1	54	Show that $V_n(\mathbb{C})$ is a complex inner product space with the inner product defined by $(u, v) = x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n}$ where $u = (x_1, x_2, \ldots, x_n)$ and $v = (y_1, y_2, \ldots, y_n)$.	7m

4	55	If T in $A_F(V)$ has a minimal polynomial $p(x) = q(x)^e$, where $q(x)$, is a monic, irreducible polynomial in $F[x]$ then basis of V over F can be found in which the matrix of T is of the form $\begin{pmatrix} c(q(x)^{e_1}) & & & \\ & c(q(x)^{e_2}) & & \\ & & \ddots & \\ & & & c(q(x)^{e_r}) \end{pmatrix}$ where $e = e_1 \geq e_2 \geq \cdots \geq e_r$.	7m
4	56	If V is finite dimensional over F, then prove that $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial for T is not zero.	7m
4	57	If V is n-dimensional vector space over F and if $T \in A(V)$ has all its characteristic roots in F then prove that T satisfies a polynomial of degree n over F.	7m
1	58	State and prove the Gram - Schmidt orthogonalization process.	8m
1	59	State and prove Cauchy - Schwartz Inequality.	8m
3	60	State and prove Spectral theorem.	8m
3	61	Show that a bilinear form f on V is symmetric if and only if every matrix that represents f is symmetric.	8m

4	62	Suppose that $V = V_1 \oplus V_2$, where V_1 and V_2 are subspace of V invariant under T. Let T_1 and T_2 be the linear transformation induced by T on V_1 and V_2 with the minimal polynomial $P_1(x)$ and $P_2(x)$ respectively. Then prove that the minimal polynomial for T over F is the least common multiple of $P_1(x)$ and $P_2(x)$.	8m
4	63	For each $i = 1, 2, ..., k$ $u_i \neq (0)$ and $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$, then prove that the minimal polynomial of T_i is $q_i(x)_{1i}$	8m
2	64	Let V be an inner product space and let N be a normal transformation on V. Then prove that the following statements are true: i). $\|N(v)\| = \|N^*(v)\|$ $\forall v \in V$ ii). $(N - cI)$ is normal, for every $c \in F$ iii). If λ_1 and λ_2 are distinct eigenvector of N with corresponding eigen values v_1 and v_2 then v_1 and v_2 are orthogonal.	10m
3	65	Let $X = [\alpha_1, \alpha_2, \ldots, \alpha_m]^T$ denote the coordinate matrix of a vector $u \in U$ relative to the basis $\mathcal{A}$ of U and let $Y = [\beta_1, \beta_2, \ldots, \beta_n]^T$ be the coordinate matrix of a vector $v \in V$ relative to the basis $\mathcal{B}$ of V. If $A = [a_{ij}]_{m\times n}$ then show that A is the matrix of the bilinear form f relative to $\mathcal{A}$ and $\mathcal{B}$ if and only if the equation $f(u, v) = X^T AY$ is satisfied for all choice $u \in U$, $v \in V$.	10m
3	66	State and prove Sylvester's law of inertia.	14m

Q.P Code: 16MSMTAS05

St. Philomena's College (Autonomous) Mysore

I Semester M.Sc. Makeup Examination September 2018

Subject: MATHEMATICS

Title: Linear Algebra (SC)

Time: 3 Hours | Max Marks: 70

PART –A

7×2=14

Answer the following:

1. a. If $V = R[x]$, the set of all polynomials over R and $W = \{f(x) \in V | f(x) = f(1-x)\}$ then prove that W is a subspace of V.

b. Prove that W is a subspace of V if and only if $L(W) = W$.

c. If V is an inner product pace then prove that $|(u, v)| \leq \|u\| \cdot \|v\| \; \forall \; u, v \in V$.

d. Solve by Crammer's rule $x_1 + 2x_2 + 3x_3 = -5$; $2x_1 + x_2 + x_3 = -7$; $x_1 + x_2 + x_3 = 0$.

e. If T is Hermitian then prove that all of its characteristic roots are real numbers.

f. If T is a linear transformation and $vT, vT) = (v, v) \; \forall \; v \in V$, then prove that T is unitary.

g. If A is a symmetric matrix then prove that any two eigen vectors form different eigen spaces are orthogonal.

PART – B

Answer the following:

2. a.i. State and prove fundamental theorem of homomorphism for vector spaces.

ii. Show that the vectors $v_1 = (1, 1, 2, 4)$, $v_2 = (2, -1, -5, 2)$, $v_3 = (1, -1, -4, 0)$ and $v_4 = (2, 1, 1, 6)$ are linearly dependent in $R^4(R)$.

iii. If W is a subspace of finite dimensional vector space V over F then prove that there exists a subspace W^1 of V such that $V = W \oplus W^1$. 5+5+4

OR

PTO

b.i. Let W be a subspace of a finite dimensional vector space V over F. Then prove that

$$\dim\left(\frac{V}{W}\right) = \dim V - \dim W.$$

ii. If S is a finite subset of a vector space V such that $V = L[S]$ then prove that there exists a subset of S which is a basis of V.

iii. (1, 1, 1) is linearly independent in $R^3(R)$. Extend it to form a basis of R^3. 6+5+3

3. a.i. If two rows of a matrix A are equal then prove that determinant of A is 0.

ii. If T is a linear transformation on an n-dimensional vector space V, then prove that characteristic and minimal polynomial for T have the same roots.

iii. Obtain the eigen values, eigen vectors and eigen spaces of $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. 5+5+4

OR

b.i. If $\{w_1, w_2, \ldots\ldots, w_m\}$ is an orthonormal set in V, then prove that

$$\sum_{i=1}^{m} |(wi, v)|^2 \leq \|v\|^2 \text{ for all } v \in V.$$

ii. Obtain an orthonormal basis with respect to the standard inner product for the subspace of R^3 generated by (1, 0, 3) and (2, 1, 1).

iii. Let S be an orthogonal set of non-zero vectors in an inner product space V. Then prove that S is a linearly independent set. 7+4+3

4. a.i. If V and W are vectors spaces over F of dimension m and n respectively then prove that dimension of $Hom(V, W$ is mn.

ii. Let A be a Hermitian matrix, then prove that there exists a unitary matrix P such that $A^1 = P^* AP$ is a diagonal matrix. 8+6

OR

PTO

b.i. If V is an n dimensional vector space over F and if $T \in A(V)$ has all its roots in F then prove that T satisfies a polynomial of degree n over F.

ii. State and prove Sylvester law of Inertia. **4+10**

5. a.i If $W \subset V$ is invariant under T, then prove that T induces a linear transformation $\tilde{T}$ on $\frac{V}{W}$ defined by $(v+W)\tilde{T} = vT + W$. if T satisfies the polynomial $q(x) \in F(x)$ then so does $\tilde{T}$. Further if $P_1(x)$ is the minimal polynomial for $\tilde{T}$ over F and if

ii. Prove that the relation similarity is an equivalence relation.

iii. Find the Jordan form of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$. **7+3+4**

OR

b.i. If $T \in A(V)$ has all its characteristic roots in F, then prove that there is a basis in which the matrix of T is triangular.

ii. Suppose that $V = V_1 \oplus V_2$, where V_1 and V_2 subspaces of V invariant under T. Let T_1 and T_2 be the linear transformation induced by T on V_1 and V_2 respectively. If the minimal polynomial T_1 over F is $P_1(x)$ and that of T_2 is $P_2(x)$, then prove that **7+7**

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