St.Philomena's College (Autonomus), Mysore

PG Department of Mathematics

Question Bank (Revised Curriculum 2018 onwards)

Second Year - Fourth Semester (2018 -20 Batch)

Course Title (Paper Title): Measure and Integration Q.P.Code-57301

Unit	S.No	Question	Marks
1	1	Give an example of of an algebra which is not Sigma algebra	2m
1	2	Give an example of a set which is both F_σ *and* G_δ set.	2m
1	3	Define a Borel set.	2m
1	4	Give an example of an uncountable set with measure zero.	2m
1	5	Give an example of a continuous function g and a measurable function h such that $h \circ g$ is not measurable.	2m
1	6	If outer measure of of a set is zero, then show that it is measurable.	2m
1	7	Construct an open set O such that $\mathbb{Q} \subset O$ *and* $m^*(O) \leq 1$	2m
1	8	Let f be a nonnegative measurable function show that $\int f = 0$ implies $f = 0$ almost everywhere.	2m
1	9	Show that the set of all transcendental number measurable set.	2m
2	10	Give an example of a signed measure which is not a measure.	2m
2	11	Which measurable subsets of $\mathbb{R}$ are connected ? Justify.	2m

1	12	If $m^*(A) > 0$ is it possible to find and open interval contained in A ? justify your answer.	2m
2	13	Define a complete measure space with an example.	2m
2	14	If f and g are bounded ,measurable functions defined on a set E of finite measure and $f \leq g$ a.e. then show that $\int_E f = \int_E g$	2m
2	15	Let f be a nonnegative measurable function show that $\int f = 0 \; implies \; f = 0 \; a.e.$	2m
2	16	Define simple function and give an example	2m
3	17	If f is absolutely continuous on $[a, b]$ then show that $E = \{x \in [a, b]/f' > 0\}$ is a measurable set.	2m
3	18	Define a signed measure and give an example.	2m
3	19	Show that the Hahn- decomposition need not be unique.	2m
3	20	Give an example of a continuous function which is not of bounded variation.	2m
4	21	Show that the characteristic function χ_A is measurable if and only if A is measurable.	2m
4	22	Give an example of a signed measure which is not a measure.	2m
4	23	Show that $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \chi_B$.	2m
4	24	Define positive set and negative set with respect to a signed measure.	2m
4	25	Show that if$\gamma_1 \; and \; \gamma_2 \; are$ singular with respect to μ then so is$c_1\gamma_1 + c_2\gamma_2$.	2m

2	26	Prove or disprove 'Any subset of a mesurable set is measurable'.	4m
2	27	If $f(x) = g(x)$, almost everywhere on E , prove that $\int_E f(x)dx = \int_E g(x)dx.$	4m
2	28	If $f(x)$ is measurable on E then prove that $(f(x))^2$ is also measurable on E	4m
2	30	If $f_1 and f_2$ are measurable on E ,then prove that $f_1 f_2$ is measurable on E	4m
3	31	If $\{f_n\}$ is a sequence of non negative measurable function which is defined on a measurable set E.Show that $\int_E lim f_n = lim \int_E f_n$	4m
3	32	If f and G are bounded measurable function on a set E of finite measure prove that$\int_E f + g = \int_E f + \int_E g.$	4m
4	33	Prove that f is bounded and measurable on E if and only if f is Lebesgue integrable on E.	4m
4	34	If f is integrable on on [a , b] and $\int_a^x f(t)dt = 0 \ \forall x \in [a, b]$,show that $f(t) = 0 \ almost \ everywhere \ on \ [a, b]$	4m
1	35	Prove that if $f(x)$ is measurable on E ,then for any real constants c, $f(x) + c$ is measurable.	5m
1	36	If $f_1 \ and \ f_2$ are measurable on E ,then prove that $f_1 + f_2$ is measurable on E.	5m
3	37	If f and G are bounded measurable function on a set E of finite measure prove that$\int_E f + g = \int_E f + \int_E g$	6m

1	38	Prove that $\int_E cf(x) = c\int_E f(x)$and $\int_E cdx = c\ m(E)$for any constant c.	7m
1	39	If $E = E_1 \cup E_2, where\ E_1\ and\ E_2\ are\ \ disjoint$ then $\int_E f(x)dx = \int_{E_1} f(x)dx + \int_{E_2} f(x)dx$	7m
1	40	State and prove Vitali covering lemma.	7m
1	41	Show that function f is of bounded variation on [a , b] if and only if it is a difference of two monotonic function	7m
3	42	If $f(x)\ and\ g(x)$ are bounded and measurable on E then $f(x)g(x)$ is Lebesgue integrable on E	7m
3	43	If $f(x)$ is bounded and measurable on E then prove that $\|f(x)\|$ is Lebesgue integrable on E.	7m
4	44	Prove that a function is Riemann Integrable in [a,b] if and only if the set of discontinuity of $f(x)$ in [a,b] has measure zero.	7m
4	45	If $f(x)$ is continuous almost everywhere in [a,b],prove that $f(x)$ is Lebesgue Integrable on [a,b]	7m
4	46	Prove that if $f(x)$ and $g(x)$ are of bounded and measure on E then $\int_E \{f(x) - g(x)\}dx = \int_E f(x)dx - \int_E g(x)dx$.	7m
4	47	Prove that $\int_{E_1 \cup E_2} f(x)dx = \int_{E_1} f(x)dx + \int_{E_2} f(x)dx - \int_{E_1 \cap E_2} f(x)dx$	7m
2	48	Prove that if $f(x)$ and $g(x)$ are of bounded and measure on E then $\int_E \{f(x) - g(x)\}dx = \int_E f(x)dx - \int_E g(x)dx$. Prove that $\int_{E_1 \cup E_2} f(x)dx = \int_{E_1} f(x)dx + \int_{E_2} f(x)dx - \int_{E_1 \cap E_2} f(x)dx$	7m

2	49	Prove that $\int_{E_1\cup E_2} f(x)dx = \int_{E_1} f(x)dx + \int_{E_2} f(x)dx - \int_{E_1\cap E_2} f(x)dx$	7m
2	50	If $\{f_n\}$ *is a* sequence of measurable functions on E then prove that $\overline{\lim} lim_{n\to\infty} f_n(x)$ *and* $\underline{\lim}_{n\to\infty} f_n(x)$ is measurable on E.	8m
2	51	If S is non measurable set ,consider the characterstic function $\chi(x) = 1\ if\ x \in S, 0\ otherwise.$ Prove that $\chi(x)$ is non measurable.	10m
3	52	If f is riemann integrable in $[a\ ,b]$ then prove that it is Lebesgue integrable in [a,b].Does the converse hold? Justify.	10m
3	53	State and prove Hahn Decomposition theorem. give an example to show that the decomposition need not be unique	10m
4	54	Define absolutely continuous function. If f is absolutely continuous on then show that f is of bounded variation. Also show that the product of two absolutely continuous function is absolutely continuous	10m
4	55	State and prove Radon- Nikodym theorem	14m

Model Question Paper

Q.P Code: 16MSMTDH01

St. Philomena's College (Autonomous) Mysore

IV Semester M.Sc. Makeup Examination : August - 2018

Subject: MATHEMATICS

Title: Measure theory and Integration

Time: 3 Hours **Max. Marks: 70**

Instruction to the Candidates: 1) *Answer All the Questions.*

2) *All questions carry equal marks.*

PART – A

Answer the following: 7×2=14

1. a. Define a Borel set.

 b. Give an example of an uncountable set which has measure zero.

 c. Prove that the function $f(x) = \begin{cases} 0 & if\ x\ irrational \\ 1 & if\ x\ rational \end{cases}$ is Lebesgue integrable.

 d. Prove that pointwise convergence alone is not sufficient to justify that $\lim_{n \to \infty} \int_E f_n = \int_E f$.

 e. Is every function of bounded variance continuous? Justify.

 f. Define a measurable space and give an example.

 g. Show that the Hahn decomposition need not be unique.

PART – B

2. a. Prove that the collection μ of all Lebesgue measurable sets is a σ - algebra. **07**

 b. Prove that the interval (a, ∞) is measureable. Deduce that every Borel set is measurable. **07**

OR

3. a. If A is any set and $\in > 0$, then show that there is an open set G such that $A \subset G$ and $m^* G \le m^* A + \in$. Prove also that there is a G in G_δ such that $A \subset G$ and $m^* A = m^* G$. **07**

 b. Give an example of a non-measurable set. **07**

PTO

4. State and prove the following:

i) Bounded Convergence theorem 06

ii) Fatou's lemma 05

OR

5. a. If f and g are integrable over E, show that the function $f+g$ is integrable over E and $\int_E f+g = \int_E f + \int_E g$. 06

b. State and prove Lebesgue Convergence theorem. 04

c. Let f be integrable over E. Then given $\in > 0$, show that there is a simple function ϕ such that $\int_E |f-\phi| < \in$. 04

6. a. State and prove Vitali's lemma. 08

b. State and prove Chebychev's inequality. Use this inequality to prove that if f is a non-negative measurable function on E, then $\int_E f = 0$ if and only if $f = 0$ $a.e$ on E. 06

OR

7. a. If f is an integrable function on $[a, b]$ and suppose that $F(x) = \int_a^x f(t)\,dt + F(a)$, then prove that $F'(x) = f(x)$ $a.e$ on $[a, b]$. 07

b. If f is absolutely continuous on $[a, b]$ and $f'(x) = 0$ $a.e$, prove that f is continuous. 07

8. State and prove Radon – Nikodym theorem. 14

OR

9. a. Let E be a measurable set such that $0 < \gamma(E) < \infty$. Prove that there is a positive set A contained in E with $\gamma(A) > 0$. 07

b. State and prove Jordan – Decomposition theorem. 07