**St.Philomena's College (Autonomus), Mysore**

**PG Department of Mathematics**

**Question Bank (Revised Curriculum 2018 onwards)**

**First Year - Second Semester ( 2019 -21 Batch)**

**Course Title (Paper Title): Real Analysis - III** Q.P.Code-57102

| Unit | S.No | Question | Marks |
|---|---|---|---|
| 1 | 1 | Define point wise convergence with example. | 2m |
| 1 | 2 | Define uniform convergence with example. | 2m |
| 1 | 3 | Show that the sum of a series of continuous function need not be continuous. | 2m |
| 1 | 4 | Show that the limit function of a sequence of continuous function need not be continuous. | 2m |
| 1 | 5 | Test the uniform convergence of $f_n(x) = \dfrac{nx}{1+n^2x^2}$; $x \in [0,1], n = 1,2,...$ | 2m |
| 1 | 6 | Let $f_n(x) = n^2x(1-x^2)^n$, $0 \leq x \leq 1$, $n = 1,2,3,...$ then show that $\int_0^1 \left\{\lim_{n\to\infty} f_n(x)\right\} dx \neq \lim_{n\to\infty} \int_0^1 f_n(x)dx$. | 2m |
| 1 | 7 | Does point wise convergence of sequence of function implies uniform convergence? Justify. | 2m |
| 1 | 8 | Test the uniform convergence of $f_n(x) = \dfrac{x}{1+nx}$; $x \in [0,1], n = 1,2,...$ | 2m |
| 1 | 9 | Examine the uniform convergence of the series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n}} \sin\left(\dfrac{x}{n}\right)$ on a bounded set $E$ of $\mathbb{R}$ | 2m |

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| 1 | 10 | Examine the uniform convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \cos\left(\frac{x}{n}\right)$ on a bounded set $E$ of $\mathbb{R}$ | 2m |
| 2 | 11 | Find the radius of convergence of $\sum_{k=0}^{\infty} 3^k x^k;\ k > 0$ | 2m |
| 2 | 12 | Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ | 2m |
| 2 | 13 | Find the radius of convergence of $\sum_{k=0}^{\infty} \left(3^k + 5^k\right) x^k;\ k > 0$ | 2m |
| 2 | 14 | Show that $\int_0^{\infty} \frac{\cos x}{\sqrt{1+x^2}} dx$ is absolutely convergent. | 2m |
| 2 | 15 | For what value of $c$ is the improper integral $\int_0^{\infty} e^{cx} dx$ convergent? | 2m |
| 2 | 16 | Examine the convergence of the improper integral $\int_2^{\infty} \frac{x^2}{\sqrt{x^7+1}} dx,\ x \in [2, \infty)$ | 2m |
| 2 | 17 | Examine the convergence of the improper integral $\int_0^{\infty} \frac{1}{\sqrt{1+2x^2}} dx,\ x \in [2, \infty)$ | 2m |
| 2 | 18 | Examine the convergence of the improper integral $\int_1^{\infty} \frac{\sin(xt)}{t^2} dt$ | 2m |
| 2 | 19 | Define Gamma function and show that $\tau\left(\frac{1}{2}\right) = \sqrt{\pi}$ | 2m |
| 2 | 20 | Show that $\tau\left(n + \frac{1}{2}\right) = \frac{(2n)!}{4^n} \frac{\sqrt{\pi}}{n!}$ | 2m |
| 3 | 21 | Define functions of several variables. | 2m |
| 3 | 22 | Show that continuity of $f$ at a point need not imply the existence of partial derivatives of $f$ that point. | 2m |
| 3 | 23 | Define homogenous function with example. | 2m |
| 3 | 24 | Define differentiability of a function. | 2m |

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| 3 | 25 | If $u = x + y - 2$, $v = x - y + z$ and $w = x^2 + y^2 + z^2 - xyz$ find $J\left(\frac{u,v,w}{x,y,z}\right)$ | 2m |
| 3 | 26 | If $f_1(x, y) = e^x \cos y$ and $f_2(x, y) = e^x \sin y$, then show that $J\left(\frac{f_1,f_2}{x,y}\right)$ is not zero at any point of $\mathbb{R}^2$. | 2m |
| 3 | 27 | If $x = \cos\theta$ $y = \sin\theta\cos\phi$ $z = -\sin\theta\sin\phi\cos\eta$ then find $J\left(\frac{x,y,z}{\theta,\phi,\eta}\right)$. | 2m |
| 3 | 28 | Show that the function $u = x+y-z$, $v = x-y+z$ and $w = x^2+y^2+z^2-2yz$ are not independent of one another. Find a relation between them. | 2m |
| 3 | 29 | Define Jacobian of functions. | 2m |
| 4 | 30 | Express the function $f(x, y) = x^2y+3y-2$ in power of $(x-1)$ and $(y+2)$. | 2m |
| 4 | 31 | Express the function $f(x, y) = x^2+xy+y^2$ in power of $(x-1)$ and $(y-2)$. | 2m |
| 4 | 32 | Define local maximum and local minimum | 2m |
| 4 | 33 | Find the rectangle of perimeter $l$ which has the maximum area. | 2m |
| 4 | 34 | Find the derivative of $y = \phi(x)$ at $x = 1$ defined implicity by the equation $x^2y^3 + 2x^2y + 3x^4 - y = 1$. | 2m |
| 4 | 35 | Find the derivative of $y = \phi(x)$ at $x = 1$ defined implicity by the equation $e^{-x\cos y} + \sin y = e$. | 2m |
| 4 | 36 | Show that the function relation $x + y + z - xyz = 0$ defines an implicit function $z = \phi(x, y)$ near $(0, 0, 0)$. Hence Deduce $\phi_x$ and $\phi_y$ | 2m |
| 4 | 37 | Find the partial derivatives $D_1\phi(x, y)$ and $D_2\phi(x, y)$ at $x = 1$, $y = -1$ defined implicity by the equation $x^2 + y^2 + z^2 - 6 = 0$. | 2m |

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| 1 | 38 | Show that the series $\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$ is uniformly convergent on $[0,k]$ for $k>0$ and deduce that $\lim_{x\to 1} \sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3} = \sum_{n=1}^{\infty} \frac{n}{n^3+1}$ | 4m |
| 2 | 39 | Define Logarithmic function. Prove that i). $e^{\alpha \log x} = x^{\alpha}$, $\alpha \in Q$ ii). $\log x \to +\infty$ as $x \to +\infty$ iii). $\lim_{x\to\infty} \frac{\log x}{x^{\alpha}} = 0$, $\alpha > 0$ | 4m |
| 2 | 40 | Evaluate $\int_0^{\infty} e^{-xy} \frac{\sin x}{x} dx$. | 4m |
| 4 | 41 | Express the function $f(x,y) = x^2y+3y-2$ in power of $(x-1)$ and $(y+2)$. | 4m |
| 4 | 42 | Find the derivative of $y=\phi(x)$ at $x=1$ defined implicity by the equation $x^2y^3+2x^2y+3x^4-y=1$ | 4m |
| 4 | 43 | Show that $f(x,y) = 2x^4-3x^2y+y^2$ has neither a maximum nor a minimum at $(0,0)$. | 4m |
| 1 | 44 | If $\{f_n\}$ is a sequence of continuous function defined on $E$ and if $\{f_n\}$ converges uniformly to $f$ on $E$ then prove that $f$ is continuous on $E$ | 6m |
| 1 | 45 | Suppose that $K$ is compact and i). $\{f_n\}$ is a sequence of continuous function defined on $K$. ii). $\{f_n\}$ converges point wise to a continuous function $f$ on $K$. iii). $\{f_n(x)\} \geq \{f_{n+1}(x)\} : \forall x \in K; n = 1,2,3,...$then show that $\{f_n\}$ converges uniformly to $f$ on $K$. | 6m |
| 1 | 46 | Test the series $\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}$for uniform convergence. Also examine the series for term by term integrability. | 6m |
| 1 | 47 | Let $f(x) = \sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)^2}$, $x \geq 1$. Prove that $f$ is differentiable and $f$ can be obtained by differentiating the series term by term. | 6m |

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| 2 | 48 | Given a double sequence $\{a_{ij}\}$, $i = 1,2,3,\ldots$, $j = 1,2,3,\ldots$, suppose that $\sum_{j=1}^{\infty} \lvert a_{ij} \rvert = b_i$ $(i = 1,2,3,\ldots)$ and $\sum b_i$ converges. Then show that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$ | 6m |
| 2 | 49 | Check the absolute convergence of the following integral: i). $\int_0^{\infty} \frac{\cos x}{\sqrt{1+x^3}} dx$ ii). $\int_0^{\infty} e^{-x^2} dx$ | 6m |
| 3 | 50 | Let $f$ be a real valued continuous function defined on an open disk $D$ containing $(x_0, y_0)$. If $D_1 f$ and $D_2 f$ exist and are continuous at $(x_0, y_0)$ then show that $f$ is differentiable at $(x_0, y_0)$. | 6m |
| 3 | 51 | If $f(x,y) = \tan^{-1}\left(\frac{x^5+y^5}{x-y}\right)$, $x \neq y$ show that $xD_1f(x,y) + yD_2f(x,y) = 2\sin(2f(x,y))$ | 6m |
| 3 | 52 | Let $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{(x^2+y^2)}, & \text{if (x,y)} \neq (0,0) \\ 0, & \text{if (x,y)=(0,0).} \end{cases}$ then show that $D_{21}f(0,0) \neq D_{12}f(0,0)$. | 6m |
| 3 | 53 | Let $f$ be a real valued continuous function defined on a non empty open subset $E$ of $\mathbb{R}^2$. If $D_1 f$ and $D_2 f$ exist on $E$ and $D_1 f$ continuous at $(x_0, y_0) \in E$ then show that $f$ is differentiable at $(x_0, y_0)$. | 6m |
| 4 | 54 | Define critical point and saddle point of function. Examine the function $f(x,y) = y^2 + 4xy + 3x^2 + x^3$ for the extreme values. | 6m |
| 1 | 55 | State and prove the cauchy criterion for convergence of sequence of functions. | 7m |

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| 1 | 56 | State and prove Weierstrass M-test. | 7m |
| 2 | 57 | If $0 < f(x) \leq g(x)$ and $f$ and $g$ are both continuous for $a \leq x < \infty$ and if $\int_a^\infty g(x)dx$ converges then show that $\int_a^\infty f(x)dx$ converges. | 7m |
| 2 | 58 | Prove that $\beta(x, y) = \dfrac{\tau(x)\tau(y)}{\tau(x+y)}$ where $x > 0$, $y > 0$. | 7m |
| 1 | 59 | State and prove the theorem on uniform convergence and continuity. | 8m |
| 1 | 60 | Prove the existence of a real continuous function on the real line which is nowhere differentiable. | 8m |
| 1 | 61 | State and prove the theorem on uniform convergence and integration. | 8m |
| 1 | 62 | State and prove the theorem on uniform convergence and differentiation for sequence of functions. | 8m |
| 2 | 63 | Define the exponential function $e^x$ and prove that i). $e^x e^y = e^{x+y}$ ii). $(e^x)' = e^x$ iii). $e^x$ is strictly increasing iv). $\lim_{x \to \infty} e^{-x} x^n = 0$ | 8m |
| 2 | 64 | If $f$ is continuous on $[a, \infty]$ and $\lim_{x \to \infty} x^p f(x) = A$ where $p > 1$. Then prove that $\int_0^\infty f(x)dx$ convergence absolutely. | 8m |
| 3 | 65 | State and prove basic mean value theorem. | 8m |
| 3 | 66 | State and prove Euler theorem for homogeneous function. | 8m |
| 3 | 67 | State and prove Young's theorem. | 8m |

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| 3 | 68 | If a function $f : E \to R$ differentiable at $(x_0, y_0) \in E$, then show that $f$ is continuous at $(x_0, y_0)$ and has partial derivatives at $(x_0, y_0)$. Is the converse of the theorem is true? Justify. | 8m |
| 4 | 69 | Suppose $f$ is continuous and has first order partial derivatives on $E \subseteq \mathbb{R}^2$. If $f$ has a local maximum (or local minimum) at a point $(x_0, y_0) \in E$ then prove that $D_1 f(x_0, y_0) = 0$ and $D_2 f(x_0, y_0) = 0$. | 8m |
| 1 | 70 | State and prove stone-Weirstrass theorem | 10m |
| 2 | 71 | State and prove Abel's theorem. | 10m |
| 2 | 72 | Let $f(x,t)$ be defined and continuous on $a \leq t < \infty$ $A \leq x \leq B$. If $\int_a^\infty f(x,t)dt$ converges uniformly to $F(x)$ on $[A, B]$ then prove that $F(x)$ is continuous on $[A, B]$ and that $\int_A^B \left[\int_a^\infty f(x,t)dt\right] dx = \int_a^\infty \left[\int_A^B f(x,t)dx\right] dt$ (10M) | 10m |
| 4 | 73 | State and prove Taylor's theorem for the function of two variables. | 10m |
| 4 | 74 | State and prove Implicit function theorem. (10M) | 10m |

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| 4 | 75 | Suppose $f$ is a real valued function defined on an open subset $G \subseteq \mathbb{R}^2$ has continuous partial derivatives of second order and let $(x_0, y_0) \in G$ be a critical point of $f$. then prove that i). if at $(x_0, y_0)$, $D_{11}f \cdot D_{22}f - (D_{12}f)^2 > 0$ and $D_{11}f > 0$ or $D_{22}f > 0$ then $f$ has a minimum at $(x_0, y_0)$ ii). if at $(x_0, y_0)$, $D_{11}f \cdot D_{22}f - (D_{12}f)^2 > 0$ and $D_{11}f < 0$ or $D_{22}f < 0$ then $f$ has a maximum at $(x_0, y_0)$ iii). if at $(x_0, y_0)$, $D_{11}f \cdot D_{22}f - (D_{12}f)^2 < 0$ then $(x_0, y_0)$ is a saddle point of $f$ iv). if at $(x_0, y_0)$, $D_{11}f \cdot D_{22}f - (D_{12}f)^2 = 0$ then the theorem gives no information. | 10m |

Model Question Paper

**Q.P Code: 16MSMTBH02**

# St. Philomena's College (Autonomous) Mysore
## II Semester M.Sc C3 Component Final Examination April - 2017

**Subject: MATHEMATICS**

**Title: REAL ANALYSIS – III (HC)**

**Time: 3 Hours** **Max Marks: 70**

**PART –A**

**Answer the following questions:** **7x2=10**

1. a. Show that the limit function of continuous function need not be continuous.

   b. Find the radius of convergence of $\sum_{k=0}^{\infty}\left(3^{K}+5^{K}\right)x^{K}$; $K>0$.

   c. For what values of C is the improper integral $\int_{0}^{\infty} e^{cx}\,dx$ is convergent?

   d. Define gamma function and show that $\Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}$. $\Gamma\frac{1}{2}=\sqrt{\pi}$

   e. Show that continuity of $f$ at a point need not imply the existence of partial derivations of $f$ at that point.

   f. Find the rectangle of perimeter $l$ which has the maximum area.

   g. Show that the function $u=x+y-2$, $v=x-y+z$ *and* $w=x^{2}+y^{2}+z^{2}-2yz$ are not independent of are another.

**PART –B**

**Answer the following:**

2. a State and prove Weirstran – M test for uniform convergence of sequence of function. 7

   b If $\{fn\}$ is a sequence of continuous function defined on E and if $fn\to f$ uniformly on E, then prove that $f$ is continuous on E. 7

OR

3. a State and prove theorem on uniform convergence and differentiation for sequence of function. 8

   b Define the exponential function $e^{x}$ and prove 6

      a) $\left(e^{x}\right)^{1}=e^{x}$

      b) $e^{x}$ is strictly increasing

      c) $\lim_{x\to\infty} e^{-x}x^{n}=0$

**PTO**

4 a If $f$ is continuous on $[a, \infty]$ and $\lim_{x \to \infty} x^n f(x) = A$ when $n > 1$, then prove that $\int_a^\infty f(x)dx$ conveys absolutely. 5

b If g(x) is continuous and decreasing for $a \le x < \infty$ and $\lim_{x \to \infty} g(x) = 0$, then prove that $\int_a^\infty g(x)\sin x\, dx$ converges. 4

c If $g$ is continuous on $a < x \le b$, $g(x)\,(x-a)^2$ is increasing an $a < x \le b$ and if $\lim_{x \to a^+} g(x)\,(x-a)^2 = 0$ then prove that $\int_a^b g(x)\sin\left(\frac{1}{x-a}\right)dx$ converges. 5

**OR**

5 a Suppose $f(x,t)$ is continuous for $a \le t < \infty, A \le x \le B$ *and* $\int_a^\infty f(x,t)\,dt$ conveys uniformly to $F(x)$ in [A, B], then prove that $\int_A^B F(x)dx = \int_a^\infty \left[\int_A^B f(x,t)\,dx\right] dt$. 6

b Prove that $\beta(x,y) = \frac{\sqrt{x}\cdot\sqrt{y}}{\sqrt{x+y}}$, $x, y > 0$. 4

c Evaluate $\int_0^1 \log\sqrt{x}\; dx$. 4

6 a State and prove mean value theorem for function of two variables. 8

b State and prove Euler's theorem for a homogeneous function of two variables. 6

**OR**

7 a State and prove Young's theorem. 10

b If $f(x,y) = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, $x \ne y$, show that $xD_1 f(x,y) + yD_2 f(x,y) = \sin(2f(x,y))$. 4

8 State and prove implicit function theorem. OR. 14

9 a Define critical point and saddle point of a function. Examine the function $f(x, y) = y^2 + 4xy + 3x^2 + x^3$ for the extreme values. 7

b Suppose f is continuous and has first order partial derivations on $E \subseteq R^2$. If $f$ has a local maximum (or local minimum) at a point $x_0, y_0) \in E$, then prove that $D_1 f(x_0, y_0) = 0$ and $D_2 f(x_0, y_0) = 0$. Further, give an example of a function having vanishing partial derivatives but not having a local maximum or local minimum at that point. 7

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